Math 245B Lecture 9 Notes

Daniel Raban

January 28, 2019

1 The Baire Category Theorem

1.1 Statement and proof of Baire's theorem

On $\mathbb{R}$, is $\mathbb{1}_{\mathbb{Q}}$ a pointwise limit of continuous functions? We will be able to answer this question and many more.

Let (X, ρ) be a complete metric space. Recall that $A \subseteq X$ is **nowhere dense** if $(\overline{A})^o = \varnothing$.

Theorem 1.1 (Baire category theorem)**.** *Let $(U_n)_{n=1}^\infty$ be a sequence of dense, open subsets of X. Then $\bigcap_{n=1}^\infty U_n$ is still dense in X. Equivalently, X is not a countable union of nowhere dense sets.*

Remark 1.1. The 2nd statement is the same statement but taking complements of all the sets involved.

Remark 1.2. This does not hold for uncountable intersections. For example, if $X = [0, 1]$, we can define $U_x = [0, 1] \setminus \{x\}$ for each $x \in X$. Then $\bigcup_x U_x = \varnothing$.

Proof. Let $x \in X$, $\delta > 0$. We must show that $B_\delta(x) \cap \bigcap_{n=1}^\infty U_n \neq \varnothing$. First, $B_\delta(x) \cap U_1 \neq \varnothing$; this intersection is open. Pick $B_{2\delta_1}(x_1) \subseteq B_\delta(x) \cap U_1$, where $2\delta_1 \leq \delta$. Then $B_{\delta_1}(x_1) \cap U_2 \neq \varnothing$; this intersection is also open. Pick $B_{2\delta_2}(x_2) \subseteq B_{\delta_1}(x_1) \cap U_2$ such that $2\delta_2 \leq \delta_1$. Continue this recursively. The end result is we get balls $B_{\delta_i}(x_i)$ with $i \geq 1$ such that $B_{2\delta_{i+1}}(x_{i+1}) \subseteq B_{\delta_i}(x_i) \cap U_{i+1}$ and $\delta_{u+1} \leq \delta_i/2$.

This tells us that $\delta_i \leq C/2^i$ for some constant C. We also get that $B_{\delta_j}(x_j) \subseteq B_{\delta_i}(x_i)$ for all $j \geq i$. So $\rho(x_j, x_i) < \delta_i$ for all $j \geq i \geq 1$, which means that the sequence $(x_i)_{i=1}^\infty$ is Cauchy. By completeness there exists a limit $y = \lim_i x_i$. Moreover, $y \in \overline{B_{\delta_i}(x_i)} \subseteq B_{2\delta_i}(x_i) \subseteq U_i$ for all i. Similarly, $y \in B_\delta(x)$. $\square$

The same proof gives a slightly more general statement.

Theorem 1.2. *If $V \subseteq X$ is open with $V \neq \varnothing$ and $(U_n)_{n=1}^\infty$ is open such that $\overline{U_n \cap V} \supseteq V$ for all n. Then $\overline{(\bigcap_{n=1}^\infty U_n) \cap V} \supseteq V$.*

1.2 Meager and residual sets

Definition 1.1. Let X be a topological space. Then $A \subseteq X$ is of **first category**[1] (or **meager**) if it is a bountable union of nowhere dense sets. $A \subseteq X$ is of **second category** (or **non-meager**) otherwise. $A \subseteq X$ is **co-meager** (or **residual**) if A^c is a meager set.

Example 1.1. The ambient space is important. $\mathbb{N}$ is meager inside $\mathbb{R}$ but residual in $\mathbb{N}$.

Corollary 1.1. *Let (X, ρ) be a complete metric space.*

1. *If $X = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed, and $U \subseteq X$ is a nonempty open set, then there exists a nonempty open $V \subseteq U$ and $n \in \mathbb{N}$ such that $V \subseteq F_n^o$.*
2. *If $X = \bigcup_{n=1}^{\infty} A_n$, and $U \subseteq X$ is a nonempty open set, then there exists a nonempty open $V \subseteq U$ and $n \in \mathbb{N}$ such that $V \subseteq (\overline{A_n})^o$.*
3. *The collection of residual sets is closed under countable intersections.*
4. *The collection of meager sets is closed under countable unions.*

Corollary 1.2. *If $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$, then there exists an n such that $\overline{E_n} \supseteq (a, b)$ for some $a < b$.*

Proof. The given condition implies that $\mathbb{R} = \bigcup_m \{q_m\} \cup \bigcup_{n=1}^{\infty} E_n$, where $(q_m)_m$ is an enumeration of $\mathbb{Q}$. Now apply Baire category. Either some $\{q_m\}$ or some E_n is dense in some (a, b). This cannot be any of the $\{q_m\}$. $\square$

Corollary 1.3. *$\mathbb{Q}$ is not the countable intersection of dense open sets.*

1.3 The Baire-Osgood theorem

Theorem 1.3 (Baire-Osgood)**.** *Let (X, ρ) be a complete metric space, and let $(f_n)_n \subseteq C(X, \mathbb{R})$ be such that $f_n \to f$ pointwise. Let A be the set of continuity points of f. Then A is residual.*

Proof. Call the **oscillation** of f at $x \in X$

$$\omega_f(x) := \inf_{\delta > 0} \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|.$$

f is continuous at A if and only if $\omega_f = 0$. So $X \setminus A = \{x : \omega_f(x) \neq 0\} = \bigcup_{k=1}^{\infty} \{x : \omega_f \geq 1/k\}$. By the Baire category theorem, it is enough to show that $\{x : \omega_f(x) > \varepsilon\}$ is closed and nowhere dense for all $\varepsilon > 0$.

To show that this is closed, let $x \notin \{x : \omega_f(x) > \varepsilon\}$. That is, let $\omega_f(x) < \varepsilon$. Then there exist $\delta > 0$ and $\varepsilon < \varepsilon$ such that $|f(y) - f(z)| \leq \varepsilon'$ for all $y, z \in B_\delta(x)$. If $x' \in B_{\delta/2}(x)$, then

[1] This was Baire's original terminology. I think meager is a much more intuitive term, though.

$B_{\delta/2}(x') \subseteq B_\delta(x)$. Then for all $y, z \in B_{\delta/2}(x')$, we have $y, z \in B_\delta(x)$. So $|f(y) - f(z)| \subseteq \varepsilon$; i.e. $\omega_f(x') \leq \varepsilon' < \varepsilon$. So $\{x : \omega_f(x) < \varepsilon\}$ is open.

Now let's show that $\{x : \omega_f(x) > \varepsilon\}$ is nowhere dense. Suppose U is a nonempty open set. Let $E_n := \bigcap_{i,j \geq n} \{x : |f_i(x) - f_j(x)| \leq \varepsilon\}$. These are closed, and $X = \bigcup_n E_n$. Then there exists an n and a nonempty open set $V \subseteq U \cap E_n^o$ containing $B_\delta(x) \ni y, z$. Then $|f_i(y) - f(y)| \leq \varepsilon$ for all $i \geq n$. Also, $\omega_f(x) = 0$, so tjere exists $|delta' < \delta$ such that $|f_i(y) - f_i(z)| \leq \varepsilon$ for all $y, z \in B_{\delta'}(x)$. By the triangle inequality, $|f(y) - f(z)| \leq 3\varepsilon$. This is a contradiction because $|f(y) - f(z)| \geq 4\varepsilon$. □

Corollary 1.4. $\mathbb{1}_{\mathbb{Q}}$ *is not the pointwise limit of continuous funcitons.*

Proof. $\mathbb{1}_{\mathbb{Q}}$ is not continuous anywhere. □